A short note about Morozov's formula

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Abstract

The purpose of this short note, is to rewrite Morozov's formula for correlation functions over the unitary group, in a much simpler form, involving the computation of a single determinant.

1 Introduction

The main result of this paper is given in 2.13.

In [1], A. Morozov proposed a formula for correlation functions of unitary matrices with Itzykson-Zuber's type measure:

$$\langle |U_{ji}|^2 \rangle_{U(N)} := \frac{1}{I(X,Y)} \int_{U(N)} \mathrm{d}U |U_{ji}|^2 \mathrm{e}^{\mathrm{tr}\,(XU^\dagger YU)} \tag{1.1}$$

where $\mathrm{d}U$ is the Haar measure over the unitary group $U(N)$ (appropriately normalized, so that 1.4 below holds with prefactor 1), and X and Y are two given diagonal complex matrices:

$$X = \mathrm{diag}(x_1, \dots, x_N) \qquad , \qquad Y = \mathrm{diag}(y_1, \dots, y_N) \tag{1.2}$$

and the normalization factor $I(X,Y)$ is the so-called Harish-Chandra-Itzykson-Zuber integral:

$$I(X,Y) := \int_{U(N)} \mathrm{d}U \mathrm{e}^{\mathrm{tr}\,(XU^\dagger YU)} \tag{1.3}$$

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It is well-known [4, 5] that:

$$I(X,Y) = \frac{\det E}{\Delta(X)\Delta(Y)} \tag{1.4}$$

where E is the matrix with entries:

$$E_{ij} := \mathrm{e}^{x_i y_j} \tag{1.5}$$

and where $\Delta(X)$ and $\Delta(Y)$ are the Vandermonde determinants:

$$\Delta(X) := \prod_{i<j}(x_i - x_j) \qquad , \qquad \Delta(Y) := \prod_{i<j}(y_i - y_j)\ . \tag{1.6}$$

Here we shall assume that $\det E \neq 0$ and $\Delta(X) \neq 0$ and $\Delta(Y) \neq 0$.

Morozov's formula was proven in [2], from Shatashvili's formula [3]. Morozov's formula was originaly written as follows, for any arbitrary sequences of complex numbers a_i, b_j one has:

$$\sum_{i,j=1}^{N} a_i b_j \int_{U(N)} \mathrm{d}U |U_{ji}|^2 \mathrm{e}^{\mathrm{tr}\,(XU^\dagger YU)} = \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho) \mathrm{e}^{\sum x_\ell y_{\rho(\ell)}}$$
$$\sum_{n=0}^{N-1}(-1)^n \sum_{i_1<i_2<\ldots<i_{n+1}} \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n-1} & \cdots & x_{i_{n+1}}^{n-1} \\ a_{i_1} & \cdots & a_{i_{n+1}} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n} & \cdots & x_{i_{n+1}}^{n} \end{pmatrix}} \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_{\rho(i_1)} & \cdots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^{n-1} & \cdots & y_{\rho(i_{n+1})}^{n-1} \\ b_{\rho(i_1)} & \cdots & b_{\rho(i_{n+1})} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_{\rho(i_1)} & \cdots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^{n} & \cdots & y_{\rho(i_{n+1})}^{n} \end{pmatrix}}\ , \tag{1.7}$$

where S_N is the smmetric group of rank N.

2 Rewriting Morozov's formula

The purpose of this short note is to rewrite this formula in a much simpler form, in a way very similar to what was done in [2].

For any two complex numbers x and y (such that $|x| > \max |x_i|$ and $|y| > \max |y_i|$), consider the choice:

$$a_i = \frac{1}{x - x_i} = \sum_{r=0}^{\infty} \frac{x_i^r}{x^{r+1}} \qquad , \qquad b_i = \frac{1}{y - y_i} = \sum_{s=0}^{\infty} \frac{y_i^s}{y^{s+1}} \tag{2.8}$$

Introduce the Schur polynomials (corresponding to hook diagramms):

$$
\begin{aligned}
S_r(x_{i_1},\ldots,x_{i_{n+1}}) \quad &:= \quad \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n-1} & \cdots & x_{i_{n+1}}^{n-1} \\ x_{i_1}^{r} & \cdots & x_{i_{n+1}}^{r} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n} & \cdots & x_{i_{n+1}}^{n} \end{pmatrix}} \\
&= \sum_{a_1\leq a_2\leq\ldots\leq a_{r-n}} \prod_{k=1}^{r-n} x_{i_{a_k}} = \sum_{j_1+\cdots+j_{n+1}=r-n} x_{i_1}^{j_1}\cdots x_{i_{n+1}}^{j_{n+1}} \,. \qquad (2.9)
\end{aligned}
$$

The formal generating function of these Schur polynomials is:

$$
\sum_{r=0}^{\infty} \frac{1}{x^{r+1}} S_r(x_{i_1},\ldots,x_{i_{n+1}}) = \prod_{k=1}^{n+1} \frac{1}{x-x_{i_k}} \qquad (2.10)
$$

Inserting that into 1.7, one gets:

$$
\begin{aligned}
& I(X,Y)\, \sum_{i,j} \frac{1}{x-x_i}\,\frac{1}{y-y_j}\,\langle |U_{ji}|^2\rangle_{U(N)} \\
&= \int_{U(N)} \mathrm{d}U\ \ \mathrm{tr}\,\left(\frac{1}{x-X} U \frac{1}{y-Y} U^\dagger\right)\ \mathrm{e}^{\mathrm{tr}\,(XU^\dagger YU)} \\
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{r,s=0}^{\infty} \sum_{\rho\in S_N} \epsilon(\rho) \mathrm{e}^{\sum x_\ell y_{\rho(\ell)}} \\
&\qquad \sum_{n=0}^{N-1} (-1)^n \sum_{i_1<i_2<\ldots<i_{n+1}} \frac{S_r(x_{i_1},\ldots,x_{i_{n+1}})}{x^{r+1}}\ \frac{S_s(y_{\rho(i_1)},\ldots,y_{\rho(i_{n+1})})}{y^{s+1}} \\
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho) \mathrm{e}^{\sum x_\ell y_{\rho(\ell)}} \sum_{n=0}^{N-1} (-1)^n \sum_{i_1<i_2<\ldots<i_{n+1}} \prod_{k=1}^{n+1} \frac{1}{x-x_{i_k}} \prod_{l=1}^{n+1} \frac{1}{y-y_{\rho(i_l)}} \\
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho) \mathrm{e}^{\sum x_\ell y_{\rho(\ell)}} \left[1-\prod_{i=1}^{N}\left(1-\frac{1}{x-x_i}\frac{1}{y-y_{\rho(i)}}\right)\right] \\
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho) \left[\prod_{i=1}^{N} \mathrm{e}^{x_i y_{\rho(i)}} - \prod_{i=1}^{N}\left(\mathrm{e}^{x_i y_{\rho(i)}} - \frac{1}{x-x_i}\,\mathrm{e}^{x_i y_{\rho(i)}}\,\frac{1}{y-y_{\rho(i)}}\right)\right] \\
&= \frac{1}{\Delta(X)\Delta(Y)} \left[\det\left(\,\mathrm{e}^{x_i y_j}\,\right) - \det\left(\mathrm{e}^{x_i y_j} - \frac{1}{x-x_i}\,\mathrm{e}^{x_i y_j}\,\frac{1}{y-y_j}\right)\right]
\end{aligned}
$$

(2.11)

Thus, Morozov's formula can be rewritten:

$$
\int_{U(N)} \mathrm{d}U\ \ \mathrm{tr}\left(\frac{1}{x-X}U\frac{1}{y-Y}U^\dagger\right)\ \mathrm{e}^{\mathrm{tr}\,(XU^\dagger YU)} = \frac{\det E - \det\left(E-\frac{1}{x-X}E\frac{1}{y-Y}\right)}{\Delta(X)\Delta(Y)} \qquad (2.12)
$$

or:

$$\boxed{\begin{aligned}\left\langle \operatorname{tr}\left(\frac{1}{x-X}U\frac{1}{y-Y}U^\dagger\right)\right\rangle &= 1-\frac{\det\left(E-\frac{1}{x-X}E\frac{1}{y-Y}\right)}{\det E} \\ &= 1-\det\left(1-\frac{1}{x-X}E\frac{1}{y-Y}E^{-1}\right)\end{aligned}} \tag{2.13}$$

3 Concluding remarks

From that expression of Morozov's formula, it is rather easy to recover any individual correlator by taking residues:

$$\left\langle U_{ij}\, U^\dagger_{ji}\right\rangle = \operatorname*{Res}_{x\to x_i}\operatorname*{Res}_{y\to x_j}\left\langle \operatorname{tr}\left(\frac{1}{x-X}U\frac{1}{y-Y}U^\dagger\right)\right\rangle \tag{3.14}$$

Notice also that 2.13 is very similar to what was found in [2] after integration over X and Y.

Notice that by expanding the determinant in 2.13 along its last column, one can find a recursion relation relating $U(N)$ to $U(N-1)$ integrals, which is equivalent to Shatashvili's approach [3].

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